

BAD BOUNDARY BEHAVIOR IN STAR INVARIANT SUBSPACES II

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ABSTRACT. We continue our study begun in [HR11] concerning the radial growth of functions in the model spaces $(IH^2)^\perp$.

1. INTRODUCTION

Suppose $I = BS_\mu$ is an inner function with Blaschke factor B , with zeros $\{\lambda_n\}_{n \geq 1}$ in the open unit disk \mathbb{D} repeated according to multiplicity, and singular inner factor S_μ with associated positive singular measure μ on the unit circle \mathbb{T} . The following result was shown by Frostman in 1942 for Blaschke products (see [Fro42] or [CL66]) and by Ahern-Clark for general inner functions [AC71, Lemma 3].

Theorem 1.1 (Frostman, 1942; Ahern-Clark, 1971). *Let $\zeta \in \mathbb{T}$ and I be inner with $\mu(\{\zeta\}) = 0$. Then the following assertions are equivalent.*

- (1) *Every divisor of I has a radial limit of modulus one at ζ .*
- (2) *Every divisor of I has a radial limit at ζ .*
- (3) *The following condition holds*

$$(1.2) \quad \sum_{n \geq 1} \frac{1 - |\lambda_n|}{|\zeta - \lambda_n|} + \int_{\mathbb{T}} \frac{1}{|\zeta - e^{it}|} d\mu(e^{it}) < \infty.$$

Based on a stronger condition than the above, Ahern and Clark [AC70] were able to characterize “good” non-tangential boundary behavior of functions in the model spaces $(IH^2)^\perp$ of the classical Hardy space H^2 (see [Nik86] for a very complete treatment of model spaces).

Theorem 1.3 ([AC70]). *Let $I = BS_\mu$ be an inner function with zeros $\{\lambda_n\}_{n \geq 1}$ and associated singular measure μ . For $\zeta \in \mathbb{T}$, the following are equivalent:*

- (1) *Every $f \in (IH^2)^\perp$ has a radial limit at ζ .*
- (2) *The following condition holds*

$$(1.4) \quad \sum_{n \geq 1} \frac{1 - |\lambda_n|}{|\zeta - \lambda_n|^2} + \int_{\mathbb{T}} \frac{1}{|\zeta - e^{it}|^2} d\mu(e^{it}) < \infty.$$

In this paper, we will study what happens when we are somewhere in between the Frostman condition (1.2) and the Ahern-Clark condition (1.4). In order to do so we will introduce an auxiliary function. Let $\varphi : (0, +\infty) \rightarrow \mathbb{R}^+$ be a positive increasing function such that

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- (1) $x \rightarrow \frac{\varphi(x)}{x}$ is bounded,
- (2) $x \mapsto \frac{\varphi(x)}{x^2}$ is decreasing,
- (3) $\varphi(x) \asymp \varphi(x + o(x))$, $x \downarrow 0$.

Such a function φ will be called *admissible*. One can check that functions like $\varphi(x) = x^p$, $1 \leq p < 2$, and $\varphi(x) = x^p \log(1/x)$, $1 < p < 2$, are admissible. Our main result is the following.

Theorem 1.5. *Let $I = BS_\mu$ be an inner function with zeros $\{\lambda_n\}_{n \geq 1}$ and associated singular measure μ , φ an admissible function, and $\zeta \in \mathbb{T}$. If*

$$(1.6) \quad \sum_{n \geq 1} \frac{1 - |\lambda_n|}{\varphi(|\zeta - \lambda_n|)} + \int_{\mathbb{T}} \frac{1}{\varphi(|\zeta - e^{it}|)} d\mu(e^{it}) < \infty,$$

then every $f \in (IH^2)^\perp$ satisfies

$$(1.7) \quad |f(r\zeta)| \lesssim \frac{\sqrt{\varphi(1-r)}}{1-r}.$$

When $\varphi(x) = x$ then we are in the Frostman situation (1.2) and no restriction is given for the growth of f since generic functions in H^2 satisfy the growth condition

$$|f(r\zeta)| = o\left(\frac{1}{\sqrt{1-r}}\right)$$

On the other hand, when $\varphi(x) = x^2$ we reach the Ahern-Clark situation (1.4). For other φ such as $\varphi(x) = x^{3/2}$ or perhaps $\varphi(x) = x^2 \log(1/x)$ we get that even though functions in $(IH^2)^\perp$ can be poorly behaved (as in the title of this paper), the growth is controlled.

There is some history behind these types of problems. When $\varphi(x) = x^{2N+2}$, where $N = 0, 1, 2, \dots$, Ahern and Clark [AC70] showed that (1.6) is equivalent to the condition that $f^{(j)}$, $0 \leq j \leq N$, have radial limits at ζ for every $f \in (IH^2)^\perp$. When $\varphi(x) = x^p$, $p \in (1, \infty)$, Cohn [Coh86] showed that (1.6) is equivalent to the condition that every $f \in H^q \cap \overline{IH_0^q}$, where $q = p(p-1)^{-1}$, has a finite radial limit at ζ .

Why did we write this second paper? In [HR11] we discussed controlled growth of functions from $(BH^2)^\perp$, where B is a Blaschke product not satisfying the condition (1.4) of the Ahern-Clark theorem. We have a general result but stated in very different terms, and using very different techniques, than the paper here. In particular, in [HR11] we obtain two-sided estimates for the reproducing kernels which yields more precise results. The results presented here are one-sided estimates but are for general inner functions and not just Blaschke products.

2. PROOF OF THE MAIN RESULT

It is well known that $(IH^2)^\perp$ is a reproducing kernel Hilbert space with kernel function

$$k_\lambda^I(z) := \frac{1 - \overline{I(\lambda)}I(z)}{1 - \overline{\lambda}z}.$$

It suffices to prove Theorem 1.5 for $\zeta = 1$. If $\|\cdot\|$ denotes the norm in H^2 , the estimate in (1.5) follows from the following result along with the obvious estimate

$$|f(r)| \leq \|f\| \|k_r^I\|, \quad f \in (IH^2)^\perp, \quad r \in (0, 1).$$

Theorem 2.1. *Let $I = BS_\mu$ be an inner function with zeros $\{\lambda_n\}_{n \geq 1}$ and associated singular measure μ and φ be an admissible function. If*

$$(2.2) \quad \sum_{n \geq 1} \frac{1 - |\lambda_n|}{\varphi(|1 - \lambda_n|)} + \int_{\mathbb{T}} \frac{1}{\varphi(|1 - e^{it}|)} d\mu(e^{it}) < \infty,$$

then

$$(2.3) \quad \|k_r^I\|^2 \lesssim \frac{\varphi(1-r)}{(1-r)^2}.$$

Proof. Our first observation is that since $x \mapsto \varphi(x)/x$ is bounded, (2.2) implies condition (1.2). By Theorem 1.1 this implies that $\lim_{r \rightarrow 1^-} |B(r)| = \lim_{r \rightarrow 1^-} |S_\mu(r)| = 1$. Hence

$$\|k_r^I\|^2 = \frac{1 - |I(r)|^2}{1 - r^2} = \frac{1 - \exp(\log(|I(r)|^2))}{1 - r^2} = \frac{1 - \exp(\log(|B(r)|^2 + \log |S_\mu(r)|^2))}{1 - r^2},$$

and since $\log |B(r)| \rightarrow 0$ and $\log |S_\mu(r)| \rightarrow 0$ when $r \rightarrow 1$, we get

$$\begin{aligned} \|k_r^I\|^2 &= \frac{1 - \exp(\log |B(r)|^2 + \log |S_\mu(r)|^2)}{1 - r^2} \\ &= \frac{1 - \left(1 + \left(\log |B(r)|^2 + \log |S_\mu(r)|^2\right) + o\left(\log |B(r)|^2 + \log |S_\mu(r)|^2\right)\right)}{1 - r^2} \\ &\sim \frac{\log |B(r)|^{-2} + \log |S_\mu(r)|^{-2}}{1 - r^2}. \end{aligned}$$

Thus to prove the estimate in (2.3) we need to prove

$$(2.4) \quad \frac{\log |B(r)|^{-2}}{1 - r^2} \lesssim \frac{\varphi(1-r)}{(1-r)^2}$$

and

$$(2.5) \quad \frac{\log |S_\mu(r)|^{-2}}{1 - r^2} \lesssim \frac{\varphi(1-r)}{(1-r)^2}.$$

Case 1: the Blaschke product B .

First note that from the Frostman condition (1.2) we get

$$(2.6) \quad \frac{1 - |\lambda_n|}{|1 - \lambda_n|} \longrightarrow 0.$$

In particular, from a certain index n_0 on the points λ_n , $n \geq n_0$, will be pseudohyperbolically far from the radius $[0, 1)$, i.e., there is a δ such that for every $n \geq n_0$ and $r \in [0, 1)$,

$$|b_{\lambda_n}(r)| \geq \delta.$$

This implies

$$\log \frac{1}{|b_{\lambda_n}(r)|^2} \asymp 1 - |b_{\lambda_n}(r)|^2.$$

A well known calculation shows that

$$1 - |b_{\lambda_n}(r)|^2 = \frac{(1 - r^2)(1 - |\lambda_n|^2)}{|1 - r\bar{\lambda}_n|^2}.$$

Thus

$$(2.7) \quad \frac{\log |B(r)|^{-2}}{1-r^2} = \frac{1}{1-r^2} \sum_{n \geq 1} \log \frac{1}{|b_{\lambda_n}(z)|^2} \asymp \sum_{n \geq 1} \frac{1-|\lambda_n|^2}{|1-\bar{\lambda}_n r|^2}.$$

Now let $\lambda_n = r_n e^{i\theta_n}$. We need the following two easy estimates:

$$(2.8) \quad |1 - \rho e^{i\theta}|^2 \asymp (1-\rho)^2 + \theta^2, \quad \rho \approx 1, \theta \approx 0.$$

$$(2.9) \quad (|z|^2 + |w|^2)^{1/2} \asymp |z| + |w|, \quad z, w \in \mathbb{C}.$$

In particular, $|1 - \lambda_n|^2 \asymp (1 - r_n)^2 + \theta_n^2$. We now remember condition (2.6) which implies that $1 - r_n = 1 - |\lambda_n| = o(|1 - \lambda_n|) = o((1 - r_n) + \theta_n)$ so that necessarily $1 - r_n = o(\theta_n)$. Hence

$$|1 - \bar{\lambda}_n r|^2 \asymp (1 - r_n r)^2 + \theta_n^2 = (1 - r_n + r_n(1 - r))^2 + \theta_n^2 \asymp (1 - r)^2 + \theta_n^2.$$

The estimate in (2.7) yields

$$(2.10) \quad \begin{aligned} \frac{\log |B(r)|^{-2}}{1-r^2} &\asymp \sum_{n \geq 1} \frac{1-|\lambda_n|^2}{|1-\bar{\lambda}_n r|^2} \asymp \sum_{n \geq 1} \frac{1-r_n}{(1-r)^2 + \theta_n^2} \asymp \sum_{\{n: 1-r < \theta_n\}} \frac{1-r_n}{\theta_n^2} + \sum_{\{n: 1-r \geq \theta_n\}} \frac{1-r_n}{(1-r)^2} \\ &= \sum_{\{n: 1-r < \theta_n\}} \frac{1-r_n}{\theta_n^2} + \frac{1}{(1-r)^2} \sum_{\{n: 1-r \geq \theta_n\}} (1-r_n). \end{aligned}$$

Let us discuss each summand in (2.10) individually. For the first, we use the fact that φ is admissible and so $\varphi(\theta) \asymp \varphi(|1 - e^{i\theta}|)$ to get

$$\begin{aligned} \sum_{\{n: 1-r < \theta_n\}} \frac{1-r_n}{\theta_n^2} &= \sum_{\{n: 1-r < \theta_n\}} \frac{1-r_n}{\sqrt{\varphi(\theta_n) \theta_n^2} \sqrt{\varphi(\theta_n)}} \\ &\leq \underbrace{\left(\sum_{\{n: 1-r < \theta_n\}} \frac{1-r_n}{\varphi(\theta_n)} \right)^{1/2}}_{\text{bounded by assumption}} \left(\sum_{\{n: 1-r < \theta_n\}} \frac{1-r_n}{\theta_n^4 / \varphi(\theta_n)} \right)^{1/2} \\ &\lesssim \left(\sum_{\{n: 1-r < \theta_n\}} \frac{1-r_n}{\varphi(\theta_n) (\theta_n^2 / \varphi(\theta_n))^2} \right)^{1/2}. \end{aligned}$$

By assumption, $x \rightarrow \varphi(x)/x^2$ is decreasing. Hence we can bound $\theta_n^2/\varphi(\theta_n)$ below in this last sum by $(1-r)^2/\varphi(1-r)$. Hence

$$\sum_{\{n: 1-r < \theta_n\}} \frac{1-r_n}{\theta_n^2} \lesssim \frac{\varphi(1-r)}{(1-r)^2} \left(\sum_{\{n: 1-r < \theta_n\}} \frac{1-r_n}{\varphi(\theta_n)} \right)^{1/2} \lesssim \frac{\varphi(1-r)}{(1-r)^2}.$$

For the second sum in (2.10) we have

$$\begin{aligned}
\sum_{\{n:1-r \geq \theta_n\}} (1-r_n) &= \sum_{\{n:1-r \geq \theta_n\}} (1-r_n) \frac{\sqrt{\varphi(\theta_n)}}{\sqrt{\varphi(\theta_n)}} \\
&\leq \underbrace{\left(\sum_{\{n:1-r \geq \theta_n\}} \frac{(1-r_n)}{\varphi(\theta_n)} \right)^{1/2}}_{\text{bounded by assumption}} \left(\sum_{\{n:1-r \geq \theta_n\}} (1-r_n) \varphi(\theta_n) \right)^{1/2} \\
&\lesssim \sqrt{\varphi(1-r)} \left(\sum_{\{n:1-r \geq \theta_n\}} (1-r_n) \right)^{1/2},
\end{aligned}$$

where we have used the fact that φ is increasing. Dividing through the square root of the sum, this last inequality (and then squaring) implies

$$\sum_{\{n:1-r \geq \theta_n\}} (1-r_n) \lesssim \varphi(1-r).$$

This verifies (2.4).

Case 2: the singular inner factor S_μ .

This case is very similar to the first case. Indeed,

$$\frac{\log |S_\mu(r)|^{-2}}{1-r^2} = 2 \int_{\mathbb{T}} \frac{1}{|1-re^{i\theta}|^2} d\mu(e^{i\theta}) \asymp \int_{\mathbb{T}} \frac{1}{(1-r)^2 + \theta^2} d\mu(e^{i\theta})$$

where we have again used (2.8). As in the Blaschke situation we split the integral into two parts depending on which term in the denominator dominates:

$$\begin{aligned}
\frac{\log |S_\mu(r)|^{-2}}{1-r^2} &\lesssim \int_{\{\theta:1-r \leq \theta\}} \frac{1}{(1-r)^2 + \theta^2} d\mu(e^{i\theta}) + \int_{\{\theta:1-r \geq \theta\}} \frac{1}{(1-r)^2 + \theta^2} d\mu(e^{i\theta}) \\
(2.11) \quad &\asymp \int_{\{\theta:1-r \leq \theta\}} \frac{1}{\theta^2} d\mu(e^{i\theta}) + \frac{1}{(1-r)^2} \int_{\{\theta:1-r \geq \theta\}} d\mu(e^{i\theta}).
\end{aligned}$$

Let us consider the first integral.

$$\begin{aligned}
\int_{\{\theta:1-r \leq \theta\}} \frac{1}{\theta^2} d\mu(e^{i\theta}) &= \int_{\{\theta:1-r \leq \theta\}} \frac{1}{\sqrt{\varphi(\theta)}\theta^2/\sqrt{\varphi(\theta)}} d\mu(e^{i\theta}) \\
&\leq \left(\int_{\{\theta:1-r \leq \theta\}} \frac{1}{\varphi(\theta)} d\mu(e^{i\theta}) \right)^{1/2} \left(\int_{\{\theta:1-r \leq \theta\}} \frac{1}{\theta^4/\varphi(\theta)} d\mu(e^{i\theta}) \right)^{1/2}.
\end{aligned}$$

Note that $|1-e^{i\theta}| \asymp \theta$. Then using the hypothesis of admissibility we have $\varphi(\theta) \asymp \varphi(|1-e^{i\theta}|)$ and so

$$\int \frac{1}{\varphi(\theta)} d\mu(e^{i\theta}) \asymp \int \frac{1}{\varphi(|1-e^{i\theta}|)} d\mu(e^{i\theta})$$

which is bounded by assumption. Hence, by the Cauchy-Schwarz inequality,

$$\int_{\{\theta:1-r \leq \theta\}} \frac{1}{\theta^2} d\mu(e^{i\theta}) \lesssim \left(\int_{\{\theta:1-r \leq \theta\}} \frac{1}{\theta^4/\varphi(\theta)} d\mu(e^{i\theta}) \right)^{1/2} = \left(\int_{\{\theta:1-r \leq \theta\}} \frac{\varphi^2(\theta)}{\varphi(\theta)\theta^4} d\mu(e^{i\theta}) \right)^{1/2}.$$

Now using the fact that $x \rightarrow \varphi(x)/x^2$ is decreasing we obtain $\varphi^2(\theta)/\theta^4 \leq (\varphi(1-r))^2/(1-r)^4$. Hence

$$\int_{\{\theta: 1-r \leq \theta\}} \frac{1}{\theta^2} d\mu(e^{i\theta}) \lesssim \frac{\varphi(1-r)}{(1-r)^2} \left(\int_{\{\theta: 1-r \leq \theta\}} \frac{1}{\varphi(\theta)} d\mu(e^{i\theta}) \right)^{1/2} \lesssim \frac{\varphi(1-r)}{(1-r)^2}.$$

We turn to the second integral in (2.11) to get

$$\begin{aligned} \int_{\{\theta: 1-r \geq \theta\}} d\mu(e^{i\theta}) &= \int_{\{\theta: 1-r \geq \theta\}} \frac{\sqrt{\varphi(\theta)}}{\sqrt{\varphi(\theta)}} d\mu(e^{i\theta}) \\ &\leq \left(\int_{\{\theta: 1-r \geq \theta\}} \varphi(\theta) d\mu(e^{i\theta}) \right)^{1/2} \left(\int_{\{\theta: 1-r \geq \theta\}} \frac{1}{\varphi(\theta)} d\mu(e^{i\theta}) \right)^{1/2}. \end{aligned}$$

We have already seen above that the second factor above is bounded by assumption. Using the fact that φ is increasing we get

$$\int_{\{\theta: 1-r \geq \theta\}} d\mu(e^{i\theta}) \lesssim \left(\int_{\{\theta: 1-r \geq \theta\}} \varphi(\theta) d\mu(e^{i\theta}) \right)^{1/2} \leq \sqrt{\varphi(1-r)} \left(\int_{\{\theta: 1-r \geq \theta\}} d\mu(e^{i\theta}) \right)^{1/2}.$$

Dividing through by the integral (and then squaring), we obtain

$$\int_{\{\theta: 1-r \geq \theta\}} d\mu(e^{i\theta}) \lesssim \varphi(1-r),$$

which verifies (2.5). ■

3. AN EXAMPLE

The Blaschke situation was discussed in [HR11] where we obtained two-sided estimates for the reproducing kernels. It can be shown with concrete examples that the estimates from Theorem 2.1 are in general weaker than those obtained in [HR11] for Blaschke products.

Let us discuss the simplest case, in fact close enough to a Blaschke product, that a singular inner function S_μ with a discrete measure μ . Let

$$\mu = \sum_{n \geq 1} \alpha_n \delta_{\zeta_n},$$

where $\delta_{\zeta_n} \in \mathbb{T}$ and α_n are positive numbers with $\sum_n \alpha_n < \infty$ guaranteeing that μ is a finite measure on \mathbb{T} . Let us fix

$$\zeta_n = e^{i\theta_n} = e^{i/n}, \quad \alpha_n = \frac{1}{n^{1+\varepsilon}}, \quad n = 1, 2, \dots$$

Also let $\varphi(t) = t^\gamma$ which defines an admissible function for $1 < \gamma < 2$. In order to have condition (2.2) it is necessary and sufficient to have

$$\sum_n \alpha_n \frac{1}{\varphi(|1 - e^{i\theta_n}|)} \simeq \sum_n \frac{1}{n^{1+\varepsilon}} \frac{1}{\varphi(1/n)} \simeq \sum_n \frac{n^\gamma}{n^{1+\varepsilon}} = \sum_n \frac{1}{n^{1+\varepsilon-\gamma}} < \infty$$

which is equivalent to $\gamma < \varepsilon$. We suppose that

$$(3.1) \quad 1 < \varepsilon < 2.$$

By Theorem 2.1 we deduce that

$$\|k_r^I\|^2 \lesssim \frac{\varphi(1-r)}{(1-r)^2} = \left(\frac{1}{1-r} \right)^{2-\gamma}.$$

In this situation we have

$$|f(r)| \lesssim \frac{1}{(1-r)^{1-\gamma/2}}, \quad f \in (S_\mu H^2)^\perp,$$

which is slower growth than the standard estimate

$$|f(r)| \lesssim \frac{1}{(1-r)^{1/2}}, \quad f \in H^2.$$

In this situation, it is actually possible to get a double-sided estimate for the reproducing kernel: since φ is admissible, Theorem 1.1 implies that $I(r) \rightarrow \eta \in \mathbb{T}$ when $r \rightarrow 1^-$. In particular for $r \in (0, 1)$, this implies that

$$|I(r)| = \exp\left(-\sum_n \alpha_n \frac{1-r^2}{|\zeta_n - r|^2}\right) \sim 1 - \sum_n \alpha_n \frac{1-r^2}{|\zeta_n - r|^2}.$$

Let us consider the reproducing kernel of $(S_\mu H^2)^\perp$ at $r = \rho_N = 1 - 2^{-N}$. Indeed,

$$\begin{aligned} \|k_{\rho_N}^I\|^2 &= \frac{1 - |I(\rho_N)|^2}{1 - \rho_N^2} \asymp 2^N \left(1 - \exp\left(-\sum_n \alpha_n \frac{1 - \rho_N^2}{|\zeta_n - \rho_N|^2}\right)\right) \\ &\asymp 2^N \left(1 - \left(1 - \sum_n \alpha_n \frac{1/2^N}{|\zeta_n - \rho_N|^2}\right)\right) \\ &\asymp \sum_n \frac{\alpha_n}{|\zeta_n - \rho_N|^2}. \end{aligned}$$

Now using (2.8)

$$|\zeta_n - \rho_N|^2 \asymp \frac{1}{n^2} + \frac{1}{2^{2N}},$$

and so

$$\begin{aligned} \|k_{\rho_N}^I\|^2 &\asymp \sum_n \frac{\alpha_n}{1/n^2 + 1/2^{2N}} = \sum_{n \leq 2^N} \frac{\alpha_n}{1/n^2} + \sum_{n > 2^N} \frac{\alpha_n}{1/2^{2N}} \\ &\asymp \sum_{n \leq 2^N} \frac{n^2}{n^{1+\varepsilon}} + 2^{2N} \sum_{n > 2^N} \frac{1}{n^{1+\varepsilon}} \asymp 2^{(2-\varepsilon)N} \\ &= \left(\frac{1}{1 - \rho_N}\right)^{2-\varepsilon} \end{aligned}$$

or, equivalently,

$$(3.2) \quad \|k_{\rho_N}^I\| \asymp \left(\frac{1}{1 - \rho_N}\right)^{1-\varepsilon/2}$$

(the estimate extends to the whole radius). As a consequence, the estimate from Theorem 2.1 is not optimal, though it is possible to come closer to it by choosing e.g., $\varphi(t) = t^\varepsilon / \log^{1+\gamma}(1/t)$, $\gamma > 0$.

4. A LOWER ESTIMATE

We finish the paper with a construction of an $f \in (S_\mu H^2)^\perp$, with μ the discrete measure discussed in the previous section, getting close to the growth given by the norm of the reproducing kernels throughout the whole radius $(0, 1)$. As in [HR11] our construction will be based on unconditional sequences. We need to recall some material on generalized interpolation in Hardy spaces for which we refer the reader to [Nik02, Section C3]. Let $I = \prod_n I_n$ be a factorization of an inner function I into inner functions I_n , $n \in \mathbb{N}$. The sequence $\{I_n\}_{n \geq 1}$ satisfies the generalized Carleson condition, sometimes called the Carleson-Vasyunin condition, which we will write $\{I_n\}_{n \geq 1} \in (CV)$, if there is a $\delta > 0$ such that

$$(4.1) \quad |I(z)| \geq \delta \inf_{n \geq 1} |I_n(z)|, \quad z \in \mathbb{D}.$$

In the special case of a Blaschke product $B = B_\Lambda$ with simple zeros $\Lambda = \{\lambda_n\}_{n \geq 1}$ and $I_n = b_{\lambda_n}$, this is equivalent to the well-known Carleson condition $\inf_n |B_{\Lambda \setminus \{\lambda_n\}}(\lambda_n)| \geq \delta > 0$.

If $\{I_n\}_{n \geq 1} \in (CV)$ then $\{(I_n H^2)^\perp\}_{n \geq 1}$ is an unconditional basis for $(IH^2)^\perp$ meaning that every $f \in (IH^2)^\perp$ can be written uniquely as

$$f = \sum_{n \geq 1} f_n, \quad f_n \in (I_n H^2)^\perp,$$

with

$$\|f\|^2 \asymp \sum_{n \geq 1} \|f_n\|^2.$$

In our situation we have $I = S_\mu$ and

$$I_n = e^{\alpha_n \frac{z + \zeta_n}{z - \zeta_n}}.$$

The corresponding spaces $(I_n H^2)^\perp$ are known to be isometrically isomorphic to the Paley-Wiener space of analytic functions of exponential type $\alpha_n/2$ and square integrable on the real axis. In this situation a sufficient condition for (4.1) is known:

$$\sup_{n \geq 1} \sum_{k \neq n} \frac{\mu(\{\zeta_n\})\mu(\{\zeta_k\})}{|\zeta_n - \zeta_k|^2} < \infty$$

(see [Nik86, Corollary 6, p. 247]). So, since $\varepsilon > 1$ by (3.1), we have

$$\sup_{n \geq 1} \sum_{k \neq n} \frac{1/n^{1+\varepsilon} 1/k^{1+\varepsilon}}{|1/n - 1/k|^2} = \sup_{n \geq 1} \sum_{k \neq n} \frac{1/n^{\varepsilon-1} 1/k^{\varepsilon-1}}{|n - k|^2} \leq \frac{\pi^2}{3} < \infty.$$

Hence $(IH^2)^\perp$ is an ℓ^2 -sum of Paley-Wiener spaces (each of which possesses for instance the harmonic unconditional basis). In particular, picking

$$\lambda_n := r_n \zeta_n = r_n e^{i/n}, \quad r_n = 1 - \frac{1}{n},$$

the sequence $\{K_n\}_{n \geq 1}$, where

$$K_n = \frac{k_{\lambda_n}^{I_n}}{\|k_{\lambda_n}^{I_n}\|} \in (I_n H^2)^\perp,$$

is an unconditional sequence in $(IH^2)^\perp$. Observe that $\Lambda = \{\lambda_n\}_{n \geq 1}$ is *not* a Blaschke sequence. We can introduce the family of functions

$$f_\beta := \sum_{n \geq 1} \beta_n K_n$$

where $\|f_\beta\|^2 \asymp \sum_{n \geq 1} |\beta_n|^2 < \infty$. Let us estimate the norms $\|k_{\lambda_n}^{I_n}\|$. First observe that

$$\alpha_n \frac{\lambda_n + \zeta_n}{\lambda_n - \zeta_n} = \alpha_n \frac{r_n + 1}{r_n - 1} = \frac{1}{n^{1+\varepsilon}} \frac{2 - 1/n}{-1/n} = -\frac{2 - 1/n}{n^\varepsilon} \longrightarrow 0, \quad n \rightarrow \infty.$$

Hence

$$\begin{aligned} \|k_{\lambda_n}^{I_n}\|^2 &= \frac{1 - |I_n(\lambda_n)|^2}{1 - r_n^2} \asymp \frac{1 - |I_n(\lambda_n)|}{1 - r_n} = \frac{1 - \exp(\log |I_n(\lambda_n)|)}{1 - r_n} \\ &= \frac{1 - \exp\left(\alpha_n \frac{\lambda_n + \zeta_n}{\lambda_n - \zeta_n}\right)}{1 - r_n} \sim \frac{1 - (1 + \alpha_n \frac{r_n + 1}{r_n - 1})}{1 - r_n} \\ &\sim \frac{2\alpha_n}{(1 - r_n)^2}, \end{aligned}$$

so that

$$\|k_{\lambda_n}^{I_n}\| \asymp \sqrt{\frac{\alpha_n}{(1 - r_n)^2}} = \frac{\sqrt{n^{-(1+\varepsilon)}}}{1/n} = n^{1-1/2-\varepsilon/2} = n^{(1-\varepsilon)/2}.$$

Observe now that the λ_n 's belong to a Stolz domain with vertex at 1. Indeed,

$$1 - |\lambda_n| = 1 - r_n = 1/n \simeq |1 - \zeta_n| \asymp |1 - \lambda_n|$$

(this follows from (2.8)). For fixed $\beta = \{\beta_n\}_{n \geq 1}$ with $\beta_n \geq 0$ we compute

$$\operatorname{Re} f_\beta(\lambda_N) \simeq \sum_{n \geq 1} \beta_n n^{(\varepsilon-1)/2} \operatorname{Re} \frac{1 - \overline{I_n(\lambda_n)} I_n(\lambda_N)}{1 - \overline{\lambda_n} \lambda_N}.$$

We have already seen that $\mathbb{R} \ni I_n(\lambda_n) \longrightarrow 1$, $n \rightarrow \infty$, and

$$I_n(\lambda_n) \sim 1 - \alpha_n \frac{1 + r_n}{1 - r_n} \sim 1 - \frac{2}{n^\varepsilon}.$$

We have to consider

$$\alpha_n \frac{\lambda_N + \zeta_n}{\lambda_N - \zeta_n}.$$

For n or N big enough, $\operatorname{Re}(\lambda_N + \zeta_n) \asymp \operatorname{Im}(\lambda_N + \zeta_n) \asymp |\lambda_N + \zeta_n| \asymp 1$. We thus have to consider the denominator. We observe that by Lemma 2.8

$$(4.2) \quad |\lambda_N - \zeta_n| = |1 - \overline{\zeta_n} \lambda_N| \asymp (1 - r_N) + \left| \frac{1}{n} - \frac{1}{N} \right| = \frac{1}{N} + \left| \frac{1}{n} - \frac{1}{N} \right| \asymp \begin{cases} \frac{1}{n} & \text{if } n \leq N \\ \frac{1}{N} & \text{if } n > N \end{cases}$$

As a consequence,

$$\alpha_n \frac{\lambda_N + \zeta_n}{\lambda_N - \zeta_n} \longrightarrow 0, \quad n \rightarrow \infty.$$

Again:

$$I_n(\lambda_N) \sim 1 + \alpha_n \frac{\lambda_N + \zeta_n}{\lambda_N - \zeta_n}.$$

Hence

$$\begin{aligned}
1 - \overline{I_n(\lambda_n)} I_n(\lambda_N) &\sim 1 - \left(1 + \alpha_n \frac{r_n + 1}{r_n - 1}\right) \left(1 + \alpha_n \frac{\lambda_N + \zeta_n}{\lambda_N - \zeta_n}\right) \sim \alpha_n \frac{1 + r_n}{1 - r_n} + \alpha_n \frac{\zeta_n + \lambda_N}{\zeta_n - \lambda_N} \\
&= \alpha_n \left(\frac{1 + r_n}{1 - r_n} + \frac{\zeta_n + \lambda_N}{\zeta_n - \lambda_N}\right) = \alpha_n \frac{(1 + r_n)(\zeta_n - \lambda_N) + (1 - r_n)(\zeta_n + \lambda_N)}{(1 - r_n)(\zeta_n - \lambda_N)} \\
&= 2\alpha_n \frac{\zeta_n - r_n \lambda_N}{(1 - r_n)(\zeta_n - \lambda_N)} = 2\alpha_n \zeta_n \frac{1 - \overline{\zeta_n} r_n \lambda_N}{(1 - r_n)(\zeta_n - \lambda_N)} \\
&= 2\alpha_n \zeta_n \frac{1 - \overline{\lambda_n} \lambda_N}{(1 - r_n)(\zeta_n - \lambda_N)}.
\end{aligned}$$

From here we have

$$(4.3) \quad \frac{1 - \overline{I_n(\lambda_n)} I_n(\lambda_N)}{1 - \overline{\lambda_n} \lambda_N} \sim \frac{2\alpha_n \zeta_n}{(1 - r_n)(\zeta_n - \lambda_N)} = \frac{2}{n^\varepsilon} \frac{\zeta_n}{\zeta_n - \lambda_N}.$$

We claim that at least for $n \geq 2N$,

$$\left| \frac{\zeta_n}{\zeta_n - \lambda_N} \right| \asymp \operatorname{Re} \frac{\zeta_n}{\zeta_n - \lambda_N}.$$

Indeed,

$$\frac{\zeta_n}{\zeta_n - \lambda_N} = \frac{1 - \zeta_n \overline{\lambda_N}}{|\zeta_n - \lambda_N|^2},$$

so that for the claim to hold it is sufficient to check that

$$|1 - \zeta_n \overline{\lambda_N}| \asymp \operatorname{Re}(1 - \zeta_n \overline{\lambda_N})$$

for $n \geq 2N$. We have already seen in (4.2) that

$$|1 - \zeta_n \overline{\lambda_N}| \asymp \frac{1}{N}, \quad n \geq 2N.$$

Now

$$\operatorname{Re}(1 - \zeta_n \overline{\lambda_N}) = 1 - r_N \operatorname{Re}\left(e^{i(1/n - 1/N)}\right) = 1 - \left(1 - \frac{1}{N}\right) \left(\cos\left(\frac{1}{n} - \frac{1}{N}\right)\right) \asymp \frac{1}{N}, \quad n \geq 2N,$$

which proves the claim. We thus can pass in (4.3) to real parts so that for $n \geq 2N$

$$\begin{aligned}
\operatorname{Re}\left(\frac{1 - \overline{I_n(\lambda_n)} I_n(\lambda_N)}{1 - \overline{\lambda_n} \lambda_N}\right) &\sim \operatorname{Re}\left(\frac{2}{n^\varepsilon} \frac{\zeta_n}{\zeta_n - \lambda_N}\right) \sim \frac{2}{n^\varepsilon} \operatorname{Re}\left(\frac{1 - \zeta_n \overline{\lambda_N}}{|\zeta_n - \lambda_N|^2}\right) \\
&\asymp \frac{2}{n^\varepsilon} \frac{1/N}{1/n^2 + (1/n - 1/N)^2} \asymp \frac{2}{n^\varepsilon} \frac{1/N}{(1/N)^2} \\
&\asymp \frac{N}{n^\varepsilon}, \quad \text{when } n \geq 2N.
\end{aligned}$$

Hence

$$\operatorname{Re} f_\beta(\lambda_N) \gtrsim \sum_{n \geq 1} \beta_n \frac{1}{n^{(1-\varepsilon)/2}} \frac{\operatorname{Re}(1 - \zeta_n \overline{\lambda_N})}{|\zeta_n - \lambda_N|^2} \gtrsim N \sum_{n \geq 2N} \frac{\beta_n}{n^{(1+\varepsilon)/2}}.$$

Pick for instance $\beta_n = n^{-(1+\eta)/2}$, where $\eta > 0$ is arbitrary, so that obviously $\beta_n \geq 0$ and $\beta \in \ell^2$. Then

$$\operatorname{Re} f_\beta(\lambda_N) \gtrsim N \sum_{n \geq 2N} \frac{1}{n^{1+(\varepsilon+\eta)/2}} \sim N \frac{1}{N^{(\varepsilon+\eta)/2}} = N^{1-\varepsilon/2-\eta/2} \asymp \left(\frac{1}{1-|\lambda_N|} \right)^{1-\varepsilon/2-\eta/2}$$

where $\eta > 0$ is arbitrarily small. Compare this with the estimate of the reproducing kernel (3.2). With better choices of β it is of course clear that we can come closer to the maximal growth given by the reproducing kernel.

Finally, we point out that when $I(z) \mapsto 1$ when $z \rightarrow 1$ in a fixed Stolz domain, it is, in general, particularly difficult to decide whether or not a sequence of reproducing kernels for $(IH^2)^\perp$, with the parameter in a Stolz domain with vertex at 1, is an unconditional basis or not. Even when $\sup_n |I(\lambda_n)| < 1$, there is a characterization known for unconditional basis which is, in general, difficult to check.

REFERENCES

- [AC70] P. R. Ahern and D. N. Clark, *Radial limits and invariant subspaces*, Amer. J. Math. **92** (1970), 332–342.
- [AC71] ———, *Radial n th derivatives of Blaschke products*, Math. Scand. **28** (1971), 189–201.
- [CL66] E. F. Collingwood and A. J. Lohwater, *The theory of cluster sets*, Cambridge Tracts in Mathematics and Mathematical Physics, No. 56, Cambridge University Press, Cambridge, 1966.
- [Coh86] W. S. Cohn, *Radial limits and star invariant subspaces of bounded mean oscillation*, Amer. J. Math. **108** (1986), no. 3, 719–749.
- [Fro42] O. Frostman, *Sur les produits de Blaschke*, Kungl. Fysiografiska Sällskapet i Lund Förhandlingar [Proc. Roy. Physiog. Soc. Lund] **12** (1942), no. 15, 169–182.
- [HR11] A. Hartmann and W.T. Ross, *Bad boundary behavior in backward shift invariant subspaces I*, preprint (2011).
- [Nik86] N. K. Nikol'skiĭ, *Treatise on the shift operator*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 273, Springer-Verlag, Berlin, 1986, Spectral function theory, With an appendix by S. V. Hruščev [S. V. Khrushchëv] and V. V. Peller, Translated from the Russian by Jaak Peetre.
- [Nik02] N. K. Nikolski, *Operators, functions, and systems: an easy reading. Vol. 2*, Mathematical Surveys and Monographs, vol. 93, American Mathematical Society, Providence, RI, 2002, Model operators and systems, Translated from the French by Andreas Hartmann and revised by the author.

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